G. MICHAEL GUY

Eliminating Envy

It seems that a lion, a fox, and an ass participated in a joint hunt. On request, the ass divides the kill into three equal shares and invites the others to choose. Enraged, the lion eats the ass, then asks the fox to make the division. The fox piles all the kill into one great heap except for one tiny morsel. Delighted at this division, the lion asks, "Who has taught you, my very excellent fellow, the art of division?" to which the fox replies, "I learnt it from the ass, by witnessing his fate."

From one of Aesop's fables, are reported by Todd Lowry in *Archaeology of Economic Ideas* (1987, p. 130)

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Definitions

A **Cake-division procedure or protocol** is a procedure that the players can use to allocate a cake among themselves (no outside arbitrators)–so that each player has a strategy that will guarantee that player a piece with which he or she is "satisified," even in the face of collusion by others.

A cake-division procedure (for n players) will be called **proportional** if each player's strategy guarantees that player a piece of size or value of at least $1/n$ in his or her own measure. It will be called **envy-free** if each player's strategy guarantees that player a piece he or she considers to be at least tied for largest, meaning this player wouldn't rather have some other piece.

Relationship between Envy-Free and Proportional.

- For $n = 2$: Envy-Free $\Leftrightarrow$ proportional.
- For $n \geq 3$ Envy-Free $\Rightarrow$ proportional. But that's it, as we shall soon see!

Divide-and-Choose
An Envy-Free Protocol for $n = 2$
At least 5000 years old!

Step 1. Player 1 cuts the cake into 2 pieces (that he considers to be the same size).

Step 2. Player 2 chooses a piece (that she considers to be at least tied for largest).

Aside. Clearly, Player 1's strategy guarantees him a piece of size exactly 1/2 in his measure, while Player 2's strategy guarantees her a piece of size at least 1/2 in her measure.

Note: Parenthetical statements are "strategies" while the main text contains the actual "protocol rules."

The Steinhaus Proportional Procedure for n=3 Circa 1943

Step 1. Player 1 cuts the cake into three pieces (that he considers to be the same size).

Step 2. Player 2 is given the choice of either passing, i.e. doing nothing (which he does if he thinks 2 or more of the pieces are of size at least 1/3), or not passing and labeling 2 of the pieces (that he thinks are of size strictly less than 1/3) as "bad."

Step 3. If Player 2 passed in step 2, then Players 3,2 and 1, in that order, choose a piece (that they consider to be of size at least 1/3).

Aside. In this case, each player receives a piece of size at least 1/3 in his own measure. This is true of: Player 3, because he chooses first; Player 2, because he thinks either 2 or 3 pieces are that large, and so at least one of them will still be available after Player 3 chooses his piece; and Player 1, because he made all 3 pieces of size 1/3.

Step 4. If Player 2 did not pass at Step 2, then Player 3 is given the same two options that Player 2 had at Step 2. He ignores Player 2's labels.

Step 5. If player 3 passed in Step 4, the Players 2, 3 and 1, in that order, choose a piece (that they consider to be of size at least 1/3).

Aside. In this case, as before, each player receives a piece of size at least 1/3 in his own measure.

Step 6. If Player 3 did not pass at Step 4, then Player 1 is required to take a piece that both Player 2 and Player 3 labelled as "bad."

Aside. Note first that there certainly must be such a piece. At this point, Player 1 has received a piece that he thinks is of size exactly 1/3, which both Player 1 and Player 2 think is "bad.", i.e. of size strictly less than 1/3.

Step 7. The other two pieces are reassembled, and Player 2 cuts the resulting piece into two pieces (that he considers to be the same size).

Step 8. Player 3 chooses one of the two pieces (that he considers to be at least tied for largest).

Step 9. Player 2 is given the remaining piece.

Aside. This is just divide-and-choose between Players 2 and 3, which ends the protocol.

Envy-Free?

Nope.

Envy-Free Protocol for $n = 3$
(Selfridge, Conway, circa 1960)

Step 1. Player 1 cuts the cake into 3 pieces (that he considers to be the same size).

Step 2. Player 2 is given the choice of either passing (which he does if he thinks two or more pieces are tied for largest), or trimming a piece from (the largest) one of the three pieces (to create a tie for the largest). If Player 2 trimmed a piece, then the trimmings are named L, for "leftover," and set aside.

Step 3. Players 3,2 and 1, in that order, choose a piece (that they consider to be at least tied for largest) from among the 3 pieces, one of which may have been trimmed in Step 2. If Player 2 did not pass in Step 2, then he is required to choose the piece he trimmed if Player 3 did not.

Aside. Notice that only part of the cake has been allocated. This yields a partition $\{X_1, X_2, X_3, L\}$ of the cake such that $\{X_1, X_2, X_3\}$ is an envy-free partial allocation. The lack of envy is true of: Player 3, because he chooses first; Player 2, because he made at least two pieces tied for the largest, and so at least one of them will still be available after Player 3 chooses his piece; and Player 1, because he made all three pieces of size 1/3, and the trimmed one has definitely been taken by either Player 3 or Player 2.

Step 4. If Player 2 passed at Step 2, we are done. Otherwise, either Player 2 or Player 3 received the trimmed piece, and the other received an untrimmed piece. Whichever player received the *un*trimmed piece now divides L into 3 pieces (that he considers to be the same size). Call this player the "cutter" and the other the "non-cutter."

Aside. We will refer to Player 1 as having an *irrevocable advantage* over the non-cutter. The point is that, since the non-cutter received the trimmed piece, Player 1 will not envy the non-cutter, *regardless* of how L is later divided among the three.

Step 5. The three pieces into which L is divided are now chosen by the players in the order: non-cutter first; Player 1 second; cutter third. (Each chooses a piece at least tied for largest among those available to him when it is his turn to choose.)

Aside. At this point, the entire cake has been allocated. Since the non-cutter chooses his piece of L first, he experiences no envy. Player 1 does not envy the non-cutter, since he had an irrevocable advantage over him, and Player 1 does not envy the cutter, because he is choosing his piece of L before the cutter does. Finally, the cutter experiences no envy since he divided L into three equal pieces.

Complexity

The Envy-Free Protocol for $n = 4$ contains 20 steps, some of which are repeated multiple times!! For $n = 3$, the number of cuts needed is at most 5, regardless of what the measures are. For $n = 4$, the number of cuts needed can be made arbitrarily large by a suitable choice of four measures, i.e. finding four friends who are "evil" enough.

Realistic?

- Not every thing is "fluid" enough to be divided in this manner.
- You might be playing this game with Aesop's Lion.

References

(1) Chapter 14 of *For All Practical Purposes*, Sixth Edition, the Math 1060 Textbook.

(2) "An Envy-Free Cake Division Protocol" by Steven J. Brams; Alan D. Taylor appearing in *The American Mathematical Monthly*, Vol. 102, No. 1 (Jan., 1995), 9-18.